A front tracking method for a strongly coupled PDE-ODE system with moving density constraints in traffic flow

Maria Laura Delle Monache, Paola Goatin

A front tracking method for a strongly coupled PDE-ODE system with moving density constraints in traffic flow

Maria Laura Delle Monache * Paola Goatin*

Abstract

In this paper we introduce a numerical method for tracking a bus trajectory on a road network. The mathematical model taken into consideration is a strongly coupled PDE-ODE system: the PDE is a scalar hyperbolic conservation law describing the traffic flow while the ODE, that describes the bus trajectory, needs to be intended in a Carathéodory sense. The moving constraint is given by an inequality on the flux which accounts for the bottleneck created by the bus on the road. The finite volume algorithm uses a locally non-uniform moving mesh which tracks the bus position. Some numerical tests are shown to describe the behavior of the solution.

1 Introduction

The first macroscopic model for traffic flow dates back to the 1950s when Lighthill and Whitham [19] and, independently, Richards [20], proposed a fluid dynamics model to describe traffic flow on an infinite single road, using a non-linear hyperbolic partial differential equation (PDE). This model is commonly referred to as the LWR model. The Cauchy problem has then been extended to initial boundary value problem in [2]. More recently several authors proposed models that track a single vehicle moving in the vehicular traffic. In these models the single vehicle trajectory is described with an ordinary differential equation (ODE), see [4, 8, 11, 18] and references therein. Indipendently, in the transport engeneering community, different numerical methods that approximate solutions for problems with moving bottlenecks have been developed, see [12, 13]. In these works the moving constraints are replaced by a sequence of fixed ones and the discontinuity is applied at the upstream cell interface with respect to the bottleneck position. Moreover, they only deal with with triangular flux diagrams.

In this article we consider the model introduced in [15] to model the effect of urban transport systems, such as buses, in a road network. From an analytical point of view, we deal with a hyperbolic conservation law which describes the evolution of the main traffic (LWR model), an ODE which describes the bus trajectory and an inequality constraint which models the bottleneck effect created by the presence of a bus on a road. Existence of solutions to this problem for general BV data was

*INRIA Sophia Antipolis - Méditerranée, EPI OPALE, 2004, route des Lucioles - BP 93, 06902 Sophia Antipolis Cedex (France). E-mail: maria-laura.delle_monache@inria.fr, paola.goatin@inria.fr.

proved in [14]. This model can be seen as a generalization to moving constraints of the problem consisting in a scalar hyperbolic conservation law with a (fixed in space) flux constraint, introduced and studied in [1, 9, 10].
The article presents a numerical method for computing the solutions of this strongly coupled PDE-ODE systems. The results are obtained by combining a tracking algorithm in Lagrangian coordinates which uses a locally nonuniform mesh as in [21] and a tracking algorithm which reconstructs the bus position through its interaction with the density waves as in [8].

The article is organized as follows. Section 2 contains some preliminary notations and definitions. Some theoretical background and the existence theorem for Cauchy problems are introduced. In Section 3 we present a finite volume scheme with moving mesh and the tracking algorithm for the solution of the ODE. In Section 4 we present some numerical tests which show the effectiveness of our approximation.

2 Mathematical Model

We consider a slow moving large vehicle (e.g. a bus) that reduces the road capacity and generates a moving bottlenck for the surrounding traffic flow. This can be modeled with a PDE-ODE coupled system consisting in a scalar conservation law representing the traffic flow with a density constraint and an ODE describing the slower vehicle trajectory:

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ \rho(t,y(t)) \leq \alpha \rho_{\max}, & t \in \mathbb{R}^+, \\ \dot{y}(t) = \omega(\rho(t,y(t)+)), & t \in \mathbb{R}^+, \\ y(0) = y_0. \end{cases} \tag{1}$$

Above $\rho = \rho(t,x) \in [0, \rho_{\max}]$ is the scalar conserved quantity denoting the mean traffic density and $\rho_{\max}$ is the maximal density allowed on the road. The flux function $f(\rho) : [0, \rho_{\max}] \to \mathbb{R}^+$ is a strictly concave function such that $f(0) = f(\rho_{\max}) = 0$. It is given by the following flux-density relation

$$f(\rho) = \rho v(\rho),$$

where v is a smooth decreasing function denoting the mean traffic speed that in this article is set to be $v(\rho) = V(1 - \frac{\rho}{\rho_{\max}})$, with V the maximal velocity allowed on the road. $y = y(t)$ represents the slower vehicle position, which moves with a speed $\omega(\rho(t,y(t)+))$. The bus velocity w therefore depends on the downstream traffic density and is given by

$$\omega(\rho) = \begin{cases} V_b & \text{if } \rho \leq \rho^* \doteq \rho_{\max}(1 - \frac{V_b}{V}), \\ v(\rho) & \text{otherwise}, \end{cases} \tag{2}$$

that it, the slower vehicle moves with a constant speed $V_b < V$ as long as it is not slowed down by the downstream traffic conditions. If this is the case, it will move at the same speed of the main traffic.
The coefficient $\alpha \in]0,1[$ gives the reduction rate of the road capacity due to the presence of the large vehicle. For simplicity, in the following we assume that $\rho_{\max} =$

$V = 1$ so that the model becomes

$$\begin{cases} \partial_t \rho + \partial_x(\rho(1-\rho)) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ \rho(t,y(t)) \le \alpha, & t \in \mathbb{R}^+, \\ \dot{y}(t) = \omega(\rho(t,y(t)+)), & t \in \mathbb{R}^+, \\ y(0) = y_0. \end{cases} \tag{3}$$

The above model was introduced in [15], in an engineering framework, to study the effect of urban transport systems in a road network. Its analytical properties were addressed in [14].

2.1 The Riemann Problem with moving density constraint

Consider (3) with the particular choice

$$y_0 = 0 \quad \text{and} \quad \rho_0(x) = \begin{cases} \rho_L & \text{if } x < 0, \\ \rho_R & \text{if } x > 0. \end{cases} \tag{4}$$

We want to define the corresponding Riemann solver with moving density constraint. To this end, we rewrite the equations in the bus reference frame (setting $X = x - V_b t$), see Figure 1, right, and we get

$$\begin{cases} \partial_t \rho + \partial_X \left(f(\rho) - V_b \rho \right) = 0, \\ \rho(0,X) = \begin{cases} \rho_L & \text{if } X < 0, \\ \rho_R & \text{if } X > 0, \end{cases} \end{cases} \tag{5}$$

under the constraint

$$\rho(t,0) \le \alpha. \tag{6}$$

Solving problem (5), (6) is equivalent to solve (5) under the corresponding constraint on the flux

$$f(\rho(t,0)) - V_b \rho(t,0) \le f_\alpha(\rho_\alpha) - V_b \rho_\alpha \doteq F_\alpha. \tag{7}$$

where $f_\alpha(\rho) = \rho\left(1 - \frac{\rho}{\alpha}\right)$ and $\rho_\alpha = \frac{\alpha}{2}(1 - V_b)$. We are now ready to define the Riemann solver for (3), (4) following [15, §V] and [14]. Denote by $\mathcal{R}$ the standard Riemann solver, i.e. the map $(t,x) \mapsto \mathcal{R}(\rho_L, \rho_R)(x/t)$ that gives the entropy solution of the conservation equation, and let $\check{\rho}_\alpha$ and $\hat{\rho}_\alpha$ with $\check{\rho}_\alpha \le \hat{\rho}_\alpha$, be the solutions of the equation $f(\rho) = f_\alpha(\rho_\alpha) + V_b(\rho - \rho_\alpha)$, see Figure 1.

Definition 2.1 *The constrained Riemann solver $\mathcal{R}^\alpha$ for (3), (4) is defined as follows.*

1. *If $f(\mathcal{R}(\rho_L,\rho_R)(V_b)) > F_\alpha + V_b \mathcal{R}(\rho_L,\rho_R)(V_b)$, then*

$$\mathcal{R}^\alpha(\rho_L,\rho_R)(x/t) = \begin{cases} \mathcal{R}(\rho_L,\hat{\rho}_\alpha)(x/t) & \text{if } x < V_b t, \\ \mathcal{R}(\check{\rho}_\alpha,\rho_R)(x/t) & \text{if } x \ge V_b t, \end{cases} \quad \text{and} \quad y(t) = V_b t.$$

2. *If $V_b \mathcal{R}(\rho_L,\rho_R)(V_b) \le f(\mathcal{R}(\rho_L,\rho_R)(V_b)) \le F_\alpha + V_b \mathcal{R}(\rho_L,\rho_R)(V_b)$, then*

$$\mathcal{R}^\alpha(\rho_L,\rho_R) = \mathcal{R}(\rho_L,\rho_R) \quad \text{and} \quad y(t) = V_b t.$$

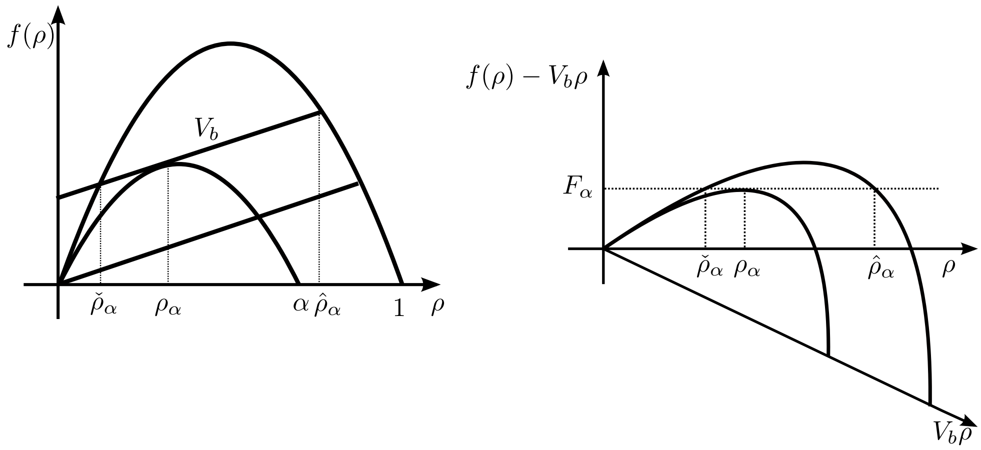

Figure 1: Flux function: Fundamental diagram, left. Bus reference frame, right.

3. If $f(\mathcal{R}(\rho_L,\rho_R)(V_b)) < V_b\mathcal{R}(\rho_L,\rho_R)(V_b)$, *then*

$$\mathcal{R}^\alpha(\rho_L,\rho_R) = \mathcal{R}(\rho_L,\rho_R) \quad \text{and} \quad y(t) = v(\rho_R)t.$$

Note that, when the constraint is enforced (point 1. in the above definition), a non-classical shock arises, which satisfies the Rankine-Hugoniot condition but violates the Lax entropy condition.

Remark 1 *The above definition is well-posed even if the classical Riemann solution* $\mathcal{R}(\rho_L,\rho_R)(x/t)$ *displays a shock at* $x = V_b t$. *In fact, due to Rankine-Hugoniot equation, we have*

$$f(\rho_L) = f(\rho_R) + V_b(\rho_L - \rho_R)$$

and hence

$$f(\rho_L) > f_\alpha(\rho_\alpha) + V_b(\rho_L - \rho_\alpha) \quad \Longleftrightarrow \quad f(\rho_R) > f_\alpha(\rho_\alpha) + V_b(\rho_R - \rho_\alpha).$$

Remark 2 *The density constraint* $\rho(t, y(t)) \le \alpha$ *does not appear explicitly in Definition 2.1, and in the following Definition 2.2. It is handled by the corresponding condition on the flux*

$$f(\rho(t,y(t))) - \omega(\rho(t,y(t)))\rho(t,y(t)) \le F_\alpha. \tag{8}$$

The corresponding density on the reduced roadway at $x = y(t)$ *is found taking the solution to the equation*

$$f(\rho_y) + \omega(\rho_y)(\rho - \rho_y) = \rho\left(1 - \frac{\rho}{\alpha}\right),$$

closer to $\rho_y \doteq \rho(t, y(t)))$, *see Figure 1.*

2.2 Cauchy Problem: existence of solutions

We briefly recall the known results for this type of problem (for details see [14]). We start giving our definition of solution.

Definition 2.2 *A couple* $(\rho, y) \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap BV(\mathbb{R}; [0, R])\right) \times \mathbf{W^{1,1}}(\mathbb{R}^+; \mathbb{R})$ *is a solution to* (3) *if*

1. ρ *is a weak solution of the conservation law, i.e., for all* $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R})$

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) \, dx = 0 \; ; \tag{9a}$$

moreover, ρ *satisfies the Kružhkov entropy conditions [17] on* $(\mathbb{R}^+ \times \mathbb{R}) \setminus \{(t, y(t)) \colon t \in \mathbb{R}^+\}$*, i.e., for every* $k \in [0, 1]$ *and for all* $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R}^+)$ *and* $\varphi(t, y(t)) = 0$*,* $t > 0$*,*

$$\begin{aligned} \int_{\mathbb{R}^+} \int_{\mathbb{R}} (|\rho - k| \partial_t \varphi + \operatorname{sgn}(\rho - k) \left(f(\rho) - f(k) \right) \partial_x \varphi) \, dx \, dt \\ + \int_{\mathbb{R}} |\rho_0 - k| \varphi(0, x) \, dx \geq 0 \; ; \end{aligned} \tag{9b}$$

2. y *is a Carathéodory solution of the ODE, i.e., for a.e.* $t \in \mathbb{R}^+$

$$y(t) = y_0 + \int_0^t \omega(\rho(s, y(s)+)) \, ds \; ; \tag{9c}$$

3. *the constraint is satisfied, in the sense that for a.e.* $t \in \mathbb{R}^+$

$$\lim_{x \to y(t)\pm} \left(f(\rho) - \omega(\rho)\rho \right)(t, x) \leq F_\alpha. \tag{9d}$$

Remark that the above traces exist because $\rho(t, \cdot) \in \mathrm{BV}(\mathbb{R})$ for all $t \in \mathbb{R}^+$.

Theorem 2.3 *Let* $\rho_0 \in BV(\mathbb{R}; [0, R])$*, then the problem* (1) *admits a solution in the sense of Definition 2.2.*

The complete proof can be found in [14]. It consists in constructing a sequence of approximate solutions via the wave-front tracking method, and prove its convergence and then check that the limit functions satisfy conditions (9a)-(9d) of Definition 2.2.

3 Numerical scheme

Our aim is to present a numerical method to compute solutions to strongly coupled constrained PDE-ODE problems. Since the solutions of the Riemann problem are known explicitely, it is natural to develop a Godunov-type method. The standard Godunov method, in principle, could be applied, however, the results produced are not correct, since it will not reproduce all the characteristics of the solutions and it fails to capture the presence of the non-classical shock. This can be overcame by applying a front tracking-capturing method which uses a Lagrangian algorithm in which the interface is tracked, such as the one in [21], together with a numerical method that tracks at each time step the slower vehicle trajectory, as in [8].

3.1 Godunov-type scheme for hyperbolic PDEs with constraint

First, we briefly recall the classical Godunov scheme and then we show how it is modified to fulfill our needs. We use the following notation: x_j^n are the grid points at time t^n with $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. A computational cell is given by $[x^n_{j-\frac{1}{2}}, x^n_{j+\frac{1}{2}}]$ where $x^n_{j\pm\frac{1}{2}}$ are the cell interfaces and $\Delta x^n_j = x^n_{j+\frac{1}{2}} - x^n_{j-\frac{1}{2}}$ is the cell size at time t^n. The Godunov scheme [16] is based on exact solutions to Riemann problems. The main idea of this method is to approximate the initial datum by a piecewise constant function, then the corresponding Riemann problems are solved exactly and a global solution is simply obtained by piecing them together. Finally, one takes the mean on the cell and proceeds by iteration. Given $U(t, x)$, the cell average of U in the cell j and at time t^n is defined as

$$U_j^n = \frac{1}{\Delta x_j^n} \int_{x^n_{j-\frac{1}{2}}}^{x^n_{j+\frac{1}{2}}} U(t^n, x) dx. \tag{10}$$

Then the Godunov scheme consists of two main steps:

1. Solve the Riemann problem at each cell interface $x^n_{j+\frac{1}{2}}$ with initial data (U_j^n, U_{j+1}^n).

2. Compute the cell averages at time t^{n+1} in each computational cell and obtain U_j^{n+1}.

We remark that waves in two neighbouring cells do not intersect before $t^{n+1} = t^n + \Delta t^n$ if the following CFL condition holds:

$$\Delta t^n \max_{j\in\mathbb{Z}} \left|\lambda^n_{j+\frac{1}{2}}\right| \leq \frac{1}{2} \min_{j\in\mathbb{Z}} \Delta x_j^n \tag{11}$$

where $\lambda^n_{j+\frac{1}{2}}$ is the wave speed of the Riemann problem solution at the interface $x_{j+\frac{1}{2}}$ at time t^n.
The classical Godunov scheme can be expressed in conservative form as

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\Big(F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)\Big), \tag{12}$$

where $F(U, V) = f(\mathcal{R}(U, V)(0))$ is the corresponding numerical flux.

Since our aim is to track the trajectory of the bus using a Lagrangian algorithm a moving mesh has to be used. In particular, we develop an algorithm which follows at each time step the bus trajectory and modifies the mesh when

$$f(\mathcal{R}(\rho_L, \rho_R)(V_b)) > F_\alpha + V_b \mathcal{R}(\rho_L, \rho_R)(V_b) \tag{13}$$

and the constraint is enforced. This factor needs to be considered when considering a Godunov scheme. In particular, if the inequality (13) is satisfied, then the solution of the Riemann solver is not the classical one and hence, the Godunov scheme cannot be applied as it is. We are going to shift grid points locally and, as a consequence, we will have a locally non-uniform mesh due to a cell interface moving with the bus trajectory. We will use the superscript *new* to indicate the quantities that are modified at time t^n with the grid. Assume that at time t^n, y^n is the bus position and $y^n \in]x^n_{m-\frac{1}{2}}, x^n_{m+\frac{1}{2}}]$, for some m. When (13) holds, the algorithm for the adaptive mesh reads as follows:

- If $\left|x^n_{m+\frac{1}{2}} - y^n\right| > \dfrac{\Delta x^n_m}{2}$, we replace the point $x^n_{m-\frac{1}{2}}$ with $x^{new}_{m-\frac{1}{2}} = y^n$ and recompute the cell averages in the cell $m-1$ from the formula

$$U^{new}_{m-1} = \frac{\Delta x^n_{m-1} U^n_{m-1} + (x^{new}_{m-\frac{1}{2}} - x^n_{m-\frac{1}{2}}) U^n_m}{\Delta x^{new}_{m-1}}, \tag{14}$$

with $\Delta x^{new}_{m-1} = x^{new}_{m-\frac{1}{2}} - x^n_{m-\frac{3}{2}}$, see Figure 2.

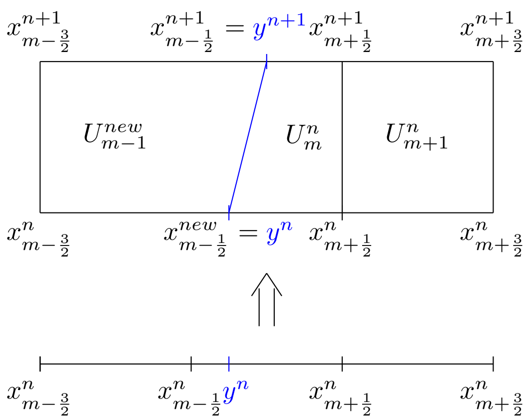

Figure 2: Local shifting of a grid point when $\left|x^n_{m+\frac{1}{2}} - y^n\right| > \dfrac{h^n_m}{2}$.

- If $\left|x^n_{m+\frac{1}{2}} - y^n\right| \leq \dfrac{\Delta x^n_m}{2}$, we adjust the location of the point $x^n_{m+\frac{1}{2}}$ such that $x^{new}_{m+\frac{1}{2}} = y^n$ and then we move the point $x^n_{m-\frac{1}{2}}$ to $x^{new}_{m-\frac{1}{2}}$ at the middle distance between $x^n_{m-\frac{3}{2}}$ and $x^{new}_{m+\frac{1}{2}}$, see Figure 3. We then compute the new cell averages in the cells m and $m+1$ from the formulas

$$U^{new}_m = \frac{(x^n_{m-\frac{1}{2}} - x^{new}_{m-\frac{1}{2}}) U^n_{m-1} + (x^{new}_{m+\frac{1}{2}} - x^n_{m+\frac{1}{2}}) U^n_m}{\Delta x^{new}_m}, \tag{15}$$

$$U^{new}_{m+1} = \frac{\Delta x^n U^n_{m+1} + (x^n_{m+\frac{1}{2}} - x^{new}_{m+\frac{1}{2}}) U^n_m}{\Delta x^{new}_{m+1}}, \tag{16}$$

with $\Delta x^{new}_m = x^{new}_{m+\frac{1}{2}} - x^{new}_{m-\frac{1}{2}}$ and $\Delta x^{new}_{m+1} = x^{new}_{m-\frac{1}{2}} - x^n_{m-\frac{3}{2}}$.

Each time the constraint is enforced the bus position follows the non-classical shock trajectory: $y^{n+1} = x^{n+1}_{m\pm\frac{1}{2}} = x^{new}_{m\pm\frac{1}{2}} + V_b \Delta t^n$. The other cell interfaces are kept unchanged.
An explicit formula for the scheme can be derived in the following way. Consider for example the finite volume cell C_{m-1} in Figure 4 (abcd). Integrate the conservation law over the finite volume:

$$\int\int_{C_{m-1}} (\partial_t U + \partial_x f(U)) dx dt = 0.$$

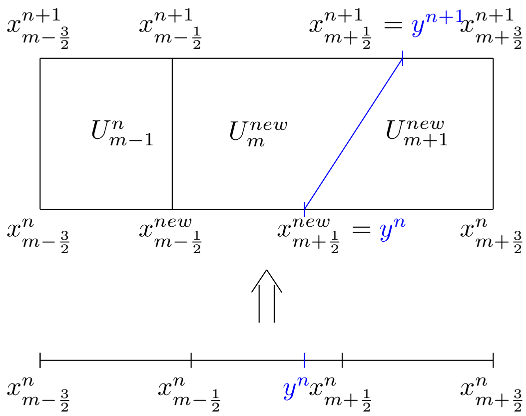

Figure 3: Local shifting of a grid point when $\left|x^n_{m+\frac{1}{2}} - y^n\right| \leq \dfrac{h^n_m}{2}$.

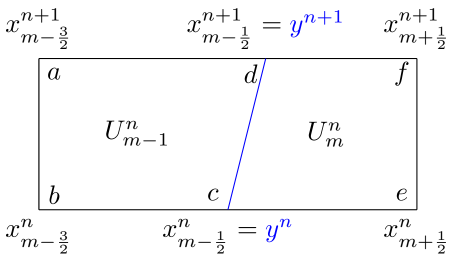

Figure 4: Non uniform finite volume cells.

From the Green's theorem we have

$$\int_{\partial C_{m-1}} f(U)dt - Udx = 0,$$

which leads to the following

$$U^{n+1}_{m-1} = \frac{\Delta x^n_{m-1}}{\Delta x^{n+1}_{m-1}} U^n_{m-1} - \frac{\Delta t^n}{\Delta x^{n+1}_{m-1}} \Big[\Big(f(\mathcal{R}(U^n_{m-1}, U^n_m)) - V_b \mathcal{R}(U^n_{m-1}, U^n_m) \Big) - f(\mathcal{R}(U^n_{m-2}, U^n_{m-1})) \Big]. \quad (17)$$

For semplicity, we introduce $\tilde{F}$ which is given by

$$\tilde{F}(U,V) = \begin{cases} F(U,V) & \text{if } f(\mathcal{R}(U,V)(V_b)) < F_\alpha + V_b \mathcal{R}(U,V)(V_b), \\ F_\alpha & \text{otherwise,} \end{cases} \quad (18)$$

so that $\tilde{F}$ corrisponds to the F_α computed in (7) when the constraint is active and to the calssical Godunov flux when the constraint is not enforced. After the mesh has been resized and adjusted as described earlier in this section, we update the cell averages for all cells with the following conservative formula:

$$U^{n+1}_j = \frac{\Delta x^{new}_j}{\Delta x^{n+1}_j} U^n_j - \frac{\Delta t^n}{\Delta x^{n+1}_j} \Big(\tilde{F}(U^n_j, U^n_{j+1}) - \tilde{F}(U^n_{j-1}, U^n_j) \Big). \quad (19)$$

3.2 Numerical method for the ODE

We expose here how to solve numerically the ODE, keeping track of the bus position. At each time t^n we determine the position y^n of the driver by studying interactions between the bus trajectory and the density waves within a fixed cell. We distinguish the two cases:

- Inequality (13) is satisfied. Then the bus moves with fixed velocity V_b and we update the bus position $y^{n+1} = y^n + V_b \Delta t^n$.
- Inequality (13) is not satisfied. In this case we implement the tracking algorithm introduced in [8]. We have to distinguish two situations: one when $y^n \in [x^n_{j-\frac{1}{2}}, x^n_j[$ and one when $y^n \in [x^n_j, x^n_{j+\frac{1}{2}}[$. In both cases, we check if the wave starting at the cell interface is a shock or a rarefaction and compute the time of interaction between the wave and the bus trajectory. In the case of the rarefaction the initial and final time of interaction is computed and the position of the bus is updated by solving explicitely an ordinary differential equation. According to the new position of the bus, the cell index is updated.

3.3 Numerical algorithm

In this section we describe in detail the algorithm which is composed of the following steps:

Algorithm 1 Algorithm for the front tracking finite volume method

Input data: Initial and boundary condition for the PDE and the ODE, m index cell corresponding to the bus position y^n.

Compute the densities at time t^{n+1} from the density values at time t^n using the Godunov flux F:

if $f(\mathcal{R}(U^n_m, U^n_m)(V_b)) > F_\alpha + V_b \mathcal{R}(U^n_m, U^n_m)(V_b)$ **then**

  if $\left| x^n_{m+\frac{1}{2}} - y^n \right| > \dfrac{\Delta x^n_m}{2}$ **then**

    $x^{new}_{m-\frac{1}{2}} = y^n$, compute the new average for U^{new}_{m-1} and update the mesh $x^{n+1}_{m-\frac{1}{2}} = x^{new}_{m-\frac{1}{2}} + V_b \Delta t^n$.

  else

    $x^{new}_{m+\frac{1}{2}} = y^n$, and place the point $x^{new}_{m-\frac{1}{2}} = \dfrac{x^n_{m-\frac{3}{2}} + x^{new}_{m+\frac{1}{2}}}{2}$. Compute the new cell averages for U^{new}_m and U^{new}_{m+1} and update the mesh $x^{n+1}_{m+\frac{1}{2}} = x^{new}_{m+\frac{1}{2}} + V_b \Delta t^n$.

  end if

end if

Compute the densities averages at time t^{n+1} using formula (19).

Compute the bus position:

if $f(\mathcal{R}(U^n_m, U^n_m)(V_b)) > F_\alpha + V_b \mathcal{R}(U^n_m, U^n_m)(V_b)$ **then**

  $y^{n+1} = V_b \Delta t^n + y^n$

else

  y^n computed with the tracking algorithm in [8]

end if

4 Numerical results

For illustration, we choose a concave fundamental diagram with the following flux function:

$$f(\rho) = \rho(1-\rho).$$

In this case the Godunov numerical flux is given by

$$F(U,V) = \begin{cases} \min\left(f(U), f(V)\right) & \text{if } U \leq V, \\ f(U) & \text{if } V < U < \rho^{\text{cr}}, \\ f^{\max} & \text{if } V < \rho^{\text{cr}} < U, \\ f(V) & \text{if } \rho^{\text{cr}} < V < U. \end{cases} \tag{20}$$

with $\rho^{\text{cr}} = 0.5$ the density at which the unique maximun of the flux function is attained, $f(\rho^{\text{cr}}) = f^{\max}$. In this section we present two numerical tests performed with the scheme previously described. We deal with a road of length 1 parametrized by the interval $[0,1]$. In both the simulations we fix $V_b = 0.3$, $\alpha = 0.6$.

1. **Case I**: We consider the following initial data

$$\rho(0,x) = 0.4, \quad y_0 = 0.5. \tag{21}$$

The solution is given by two classical shocks separated by a non-classical discontinuity, as illustrated in Figures 5 and 6.

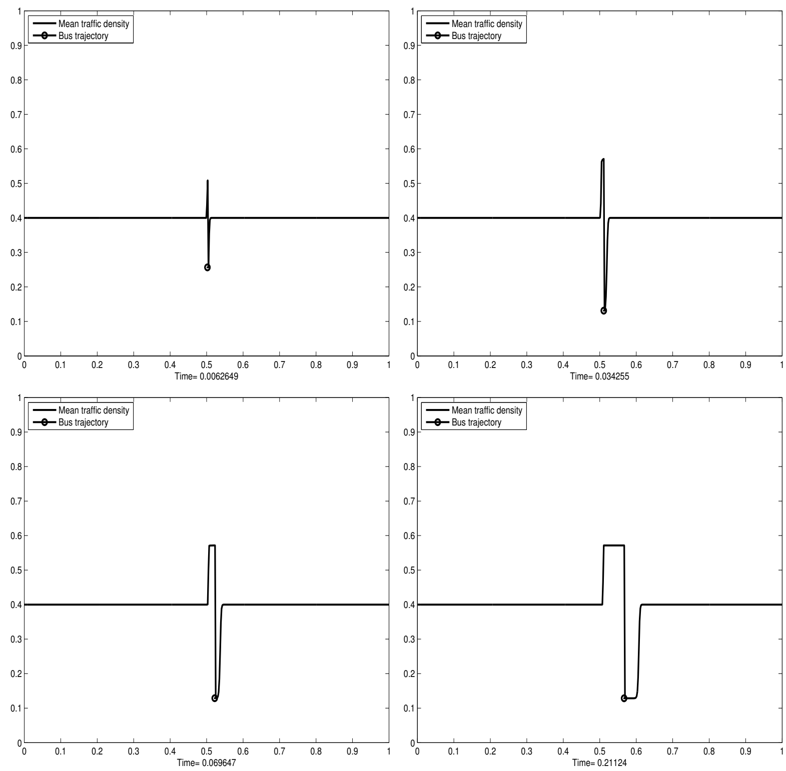

Figure 5: Density evolution at different times corresponding to initial data (21) and a mesh grid of 500 points.

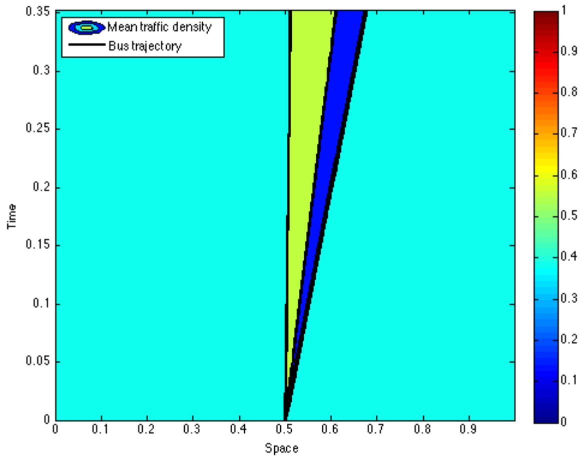

Figure 6: Density and bus trajectory in the $x-t$ plane corresponding to initial data (21) and a mesh grid of 500 points.

2. **Case II**: We consider the following initial data

$$\rho_l(0,x) = 0.8, \quad \rho_r(0,x) = 0.53, \quad y_0 = 0.5. \tag{22}$$

The values of the initial conditions create a rarefaction wave followed by a non-classical and a classical shocks on the density, as illustrated in Figures 8 and 7.

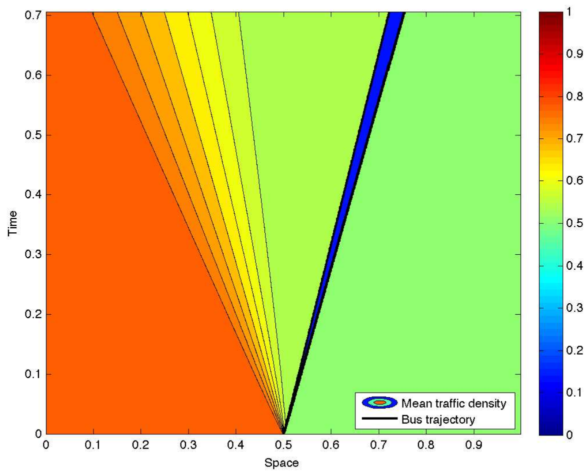

Figure 7: Density and bus trajectory in a $x-t$ plane corresponding to initial data (22) and a mesh grid of 500 points.

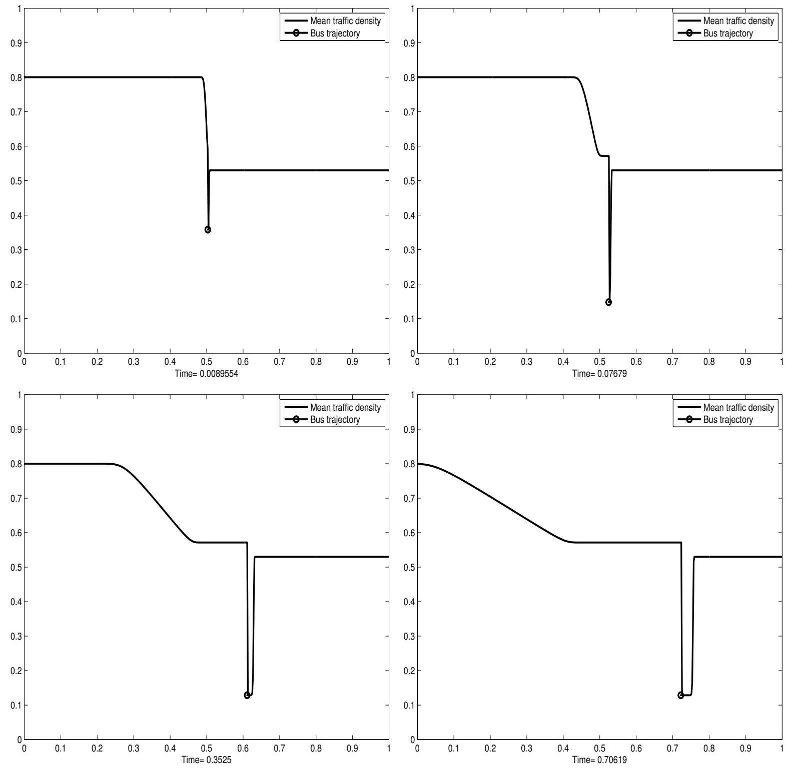

Figure 8: Evolution of the density at different times corresponding to initial data (22) and a mesh grid of 500 points.

5 Conclusions

This article introduces a numerical method for a strongly coupled PDE-ODE problem with moving density constraint which can model the presence of a moving bottleck on the roads. The PDE describes the evolution of the main traffic in time while the ODE describes the bus trajectory. The theoretical framework of the model was set up. The density is computed using a Godunov-type scheme with a locally nonuniform mesh. Then the position of the bus is recostructed determining the effects of the interactions with density waves as in [8]. Some numerical tests are presented to show the effectiveness of the scheme.

6 Acknowledgment

This research was supported by the European Research Council under the European Union's Seventh Framework Program (FP/2007-2013) / ERC Grant Agreement n. 257661 .

References

[1] B. Andreianov, P. Goatin and N. Seguin *Finite volume scheme for locally constrained conservation laws*, Numer. Math., **115** (2010), 609–645.

[2] C. Bardos, A.Y. LeRoux and J.C.. Nédélec, *First order quasilinear equations with boundary conditions*, Comm. Partial Differential Equations, **4** (1979), 1017–1034.

[3] R. Borsche, R. M. Colombo and M. Garavello, *On the coupling of systems of hyperbolic conservation laws with ordinary differential equations*, Nonlinearity, **43** (2010), 2749–2770.

[4] R. Borsche, R. M. Colombo and M. Garavello, *Mixed systems: ODEs - Balance laws*, Journal of Differential equations, **252** (2012), 2311–2338.

[5] A. Bressan, *"Hyperbolic systems of conservation laws - The one dimensional Cauchy problem,"* 1st edition, Oxford university press, 2000.

[6] A. Bressan and P. G. LeFloch, *Structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws*, Indiana University Mathematical Journal, **48** (1999), 43–84.

[7] A. Bressan and W. Shen, *Uniqueness for discontinuous ODE and conservation laws*, Nonlinear Analysis, **34** (1998), 637–652.

[8] G. Bretti and B. Piccoli, *A tracking algorithm for car paths on road networks*, SIAM Journal of Applied Dynamical Systems, **7** (2008), 510–531.

[9] C. Chalons, P. Goatin and N. Seguin, *General constrained conservation laws. Application to pedestrian flow modeling*, Netw. Heterog. Media, **8(2)** (2013), 433–463.

[10] R. M. Colombo and P. Goatin, *A well posed conservation law with variable unilateral constraint*, Journal of Differential Equations, **234** (2007), 654–675.

[11] R. M. Colombo and A. Marson, *A Hölder continuous O.D.E. related to traffic flow*, The Royal Society of Edinburgh Proceedings A, **133** (2003), 759–772.

[12] C.F. Daganzo and J.A. Laval, *On the numerical treatement of moving bottlenecks*, Transportation Research Part B **39** (2005), 31–46.

[13] C.F. Daganzo and J.A. Laval, *Moving bottlenecks: A numerical method that converges in flows*, Transportation Research Part B **39** (2005), 855–863.

[14] M. L. Delle Monache and P. Goatin, *Scalar conservation laws with moving density constraints*, INRIA Research Report, n.8119 2012, http://hal.inria.fr/hal-00745671.

[15] F. Giorgi, "Prise en compte des transports en commune de surface dans la modélisation macroscopique de l'écoulement du trafic," Ph.D thesis, Insitut National des Sciences Appliquèes de Lyon, 2002.

[16] S. K. Godunov, *A finite difference method for the numerical computation of discontinuous solutions of the equations of fluid dynamics*, Matematicheskii Sbornik, **47** (1959), 271–290.

[17] N. Kružhkov, *First order quasilinear equations with several independent variables*, Matematicheskii Sbornik, **81** (1970), 228–255.

[18] C. Lattanzio, A. Maurizi and B. Piccoli, *Moving bottlenecks in car traffic flow: a pde-ode coupled model*, SIAM Journal of Mathematical Analysis, **43** (2011), 50–67.

[19] M. J. Lighthill and G. B. Whitham, *On kinetic waves. II. Theory of traffic flows on long crowded roads*, Proceedings of the Royal Society of London Series A, **229** (1955), 317–345.

[20] P. I. Richards, *Shock waves on the highways*, Operational Research, **4** (1956), 42–51.

[21] X. Zhong, T. Y. Hou and P. G. LeFloch, *Computational Methods for propagating phase boundaries*, Journal of Computational Physics, **124** (1996), 192–216.